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The investigation for (2+1)-dimensional Eckhaus-type extension of the dispersive long wave equation

Zhenya Yan

Key Laboratory of Mathematics Mechanization, Institute of Systems Science, AMSS,
Chinese Academy of Sciences, Beijing 100080, People's Republic of China

E-mail: zyyan@mmrc.iss.ac.cn

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Abstract

The (2+1)-dimensional Eckhaus-type extension of the dispersive long wave (EEDLW) equation is investigated, which was obtained in the appropriate approximation from the basic equations of hydrodynamics. Though it has no Painlevé property, we gain an auto-Bäcklund transformation (aBT) by truncating the Laurent series expansion at $O(w^0)$. In particular, the special one of the aBT establishes a relationship between the EEDLW equation and a set of three linear partial differential equations involving the well-known heat equation. Finally many types of new exact solutions of the EEDLW equation are found from the obtained aBT and some proper ansätze, which may be useful to explain some physical phenomena.

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1. Introduction

The investigation of exact solutions to nonlinear evolution equations (NLEEs) has become an interesting subject in the field of nonlinear science [1] since the ‘soliton’ concept was first introduced in 1965 [2]. Many physical NLEEs have been derived from both theoretical and experimental work such as the KdV equation, the mKdV equation, the Burgers equation, the Boussinesq equations, the dispersive long wave equation and the (2+1)-dimensional KP equation, etc. The study of (2+1)-dimensional NLEEs, even higher dimensional NLEEs, has also attracted more attention (see [1, 6–9, 11–16, 18] for details).

The popular (1+1)-dimensional dispersive long wave equation [1, 3]

$$u_t + \eta_x + \frac{1}{2}(u^2)_x = 0 \quad \eta_t + (u\eta + u + u_{xx})_x = 0 \quad (1)$$

plays an important role in nonlinear physics, which describes the evolution of horizontal

velocity component $u(x, t)$ of water waves of height $\eta(x, t)$ propagating in both directions in an infinite narrow channel of finite constant depth. Many properties of (1) have been reported [3–5].

It is interesting to study the extensions of (1) in higher-dimensional spaces. To date, there exist two prototypical extensions of (1) to cover the situation of wide channel or open seas. In 1987, Boiti *et al* [6] presented the following (2+1)-dimensional extension related to (1)

$$u_{ty} + \eta_{xx} + \frac{1}{2}(u^2)_{xy} = 0 \quad \eta_t + (u\eta + u + u_{xy})_x = 0 \quad (2)$$

arising from the compatibility conditions $\{[T_1, T_2]\psi = 0, T_1\psi = 0\}$ of the ‘weak’ Lax pair

$$\begin{aligned} T_1\psi &= [\partial_x \partial_y - \frac{1}{2}u(x, y, t)\partial_y - \frac{1}{4}(u_y - 1 - \eta(x, y, t))]\psi = 0 \\ T_2\psi &= [\partial_t + \partial_x^2 + \gamma(x, y, t)]\psi = 0. \end{aligned} \quad (3)$$

In the one-dimensional reduction $u = u(x + y, t)$, $\eta = \eta(x + y, t)$, system (2) reduces to system (1). The Bäcklund transformation, soliton solutions and superposition were given [6, 8]. Paquin and Winternitz [7] gave the infinite-dimensional symmetry groups and a Kac–Moody–Virasoro structure of (2). It has been shown that (2) has no Painlevé property [18], even if it has a Lax pair and is solvable by the inverse spectral transformation [6].

In 1985, Eckhaus [9] presented another different two-dimensional extension of (1)

$$u_t + \eta_x + \frac{1}{2}(u^2)_x = 0 \quad (4a)$$

$$\eta_{tx} + (u\eta + u + u_{xx})_{xx} + u_{yy} = 0 \quad (4b)$$

which was obtained in the appropriate approximation from the basic equations of hydrodynamics. It is easy to see that if one makes the transformation $\{u = u(x + y, t), \eta = \eta(x + y, t) - 1\}$ or $\{u = u(x, t), \eta = (x, t)\}$, then (4) can also reduce to (1). Therefore it follows that these two systems (2) and (4) can both reduce to the same system (1) under the proper transformations. But as Boiti *et al* [6] pointed out, system (2) is different from system (4). It is shown that (4) possesses only finite-dimensional symmetry groups and is not Painlevé integrable [7].

As far as we know, no work on other properties of (4) seems to have been reported such as auto-Bäcklund transformation (aBT) and exact solutions, etc. As Flaschka *et al* [10] said, for a system that does not have the Painlevé property and does not even have the Laurent property for arbitrary singular manifold w , one may have also attempted to find a self-consistent system of equations by truncating the Laurent series expansion at $O(w^0)$ with regard to whether the system actually has the Laurent property.

In this paper, we would like to consider it by truncating the Laurent series expansion at $O(w^0)$ such that an auto-Bäcklund transformation is obtained, though (4) does not pass the Painlevé test. In particular, a special one of the aBT establishes a relationship between (4) and a linear system involving three linear partial differential equations. Moreover many types of new exact solutions of (4) are also obtained from the obtained aBT and some proper ansätze.

The rest of the paper is organized as follows: In section 2, a new auto-Bäcklund transformation of (4) is obtained by truncating the Laurent series expansion at $O(w^0)$. In particular, (4) can reduce to a linear system involving three linear partial differential equations. In section 3, some explicit and exact solutions are obtained. Finally, some conclusions and some open questions are given in section 4.

2. The auto-Bäcklund transformation

Though the system (4) does not pass the Painlevé test, it is important to derive an auto-Bäcklund transformation of (4), which is useful to seek the exact solutions of (4). Expanding both $u(x, y, t)$ and $\eta(x, y, t)$ in (4) about the same singular manifold $w(x, y, t) = 0$ [10] yields

$$u(x, y, t) = \sum_{j=0}^{\infty} u_j w^j \quad \eta(x, y, t) = \sum_{j=0}^{\infty} \eta_j w^j. \tag{5a}$$

By balancing the highest order linear term η_x and nonlinear terms $(u^2)_x$ in (4a) as well as u_{xxx} and $(u\eta)_{xx}$ in (4b), we truncate the Laurent series expansions (5a) at $O(w^0)$ as

$$u(x, y, t) = \frac{u_0}{w} + u_1 \quad \eta(x, y, t) = \frac{\eta_0}{w^2} + \frac{\eta_1}{w} + \eta_2 \tag{5b}$$

where $u_0, u_1, \eta_0, \eta_1, \eta_2, w$ are functions of (x, y, t) to be determined. The substitution of (5b) into (4) gives rise to

$$u_0 = 2\mu w_x \quad \mu = \pm 1 \quad \eta_0 = -2w_x^2 \quad \eta_1 = 2w_{xx}. \tag{5c}$$

Therefore the expression (5a) further reduces to

$$\begin{aligned} u(x, y, t) &= \frac{2\mu w_x}{w} + u_1 = 2\mu \partial_x \log[w(x, y, t)] + u_1(x, y, t) \\ \eta(x, y, t) &= -\frac{2w_x^2}{w^2} + \frac{2w_{xx}}{w} + \eta_2 = 2\partial_x^2 \log[w(x, y, t)] + \eta_2(x, y, t) \end{aligned} \tag{6}$$

where (u_1, η_2) solves (4). The substitution of (6) into (4a) leads to

$$\begin{aligned} u_t + \eta_x + \frac{1}{2}(u^2)_x &= 2\mu \partial_x \partial_t \log[w(x, y, t)] + 2\partial_x^3 \log[w(x, y, t)] + 2\partial_x (\partial_x \log[w(x, y, t)])^2 \\ &+ 2\mu \partial_x (u_1 \partial_x \log[w(x, y, t)]) + u_{1,t} + \eta_{2,x} + \frac{1}{2}(u_1^2)_x = 0. \end{aligned} \tag{7}$$

Since (u_1, η_2) satisfies (4a), with the aid of symbolic computation (Maple), we know that (7) reduces to

$$\begin{aligned} u_t + \eta_x + \frac{1}{2}(u^2)_x &= \partial_x \{ 2\mu \partial_t \log[w(x, y, t)] + 2\partial_x^2 \log[w(x, y, t)] \\ &+ 2(\partial_x \log[w(x, y, t)])^2 + 2\mu u_1 \partial_x \log[w(x, y, t)] \} \\ &= 2\mu \partial_x \left[\frac{1}{w} (w_t + \mu w_{xx} + u_1 w_x) \right] = 0. \end{aligned} \tag{8}$$

Similarly, the substitution of (6) into (4b) gives rise to

$$\begin{aligned} \eta_{tx} + (u\eta + u + u_{xx})_{xx} + u_{yy} &= \partial_x \{ 2\partial_x^2 \partial_t \log[w(x, y, t)] + 4\mu (\partial_x^2 \log[w(x, y, t)])^2 \\ &+ 4\mu (\partial_x \log[w(x, y, t)]) (\partial_x^3 \log[w(x, y, t)]) \\ &+ (2\mu \eta_2 + 2u_{1,x}) \partial_x^2 \log[w(x, y, t)] + 2\mu \eta_{2,x} \partial_x \log[w(x, y, t)] \\ &+ 2\mu \partial_x^2 \log[w(x, y, t)] + 2u_1 \partial_x^3 \log[w(x, y, t)] \\ &+ 2\mu \partial_x^4 \log[w(x, y, t)] + 2\mu \partial_y^2 \log[w(x, y, t)] \} \\ &+ \eta_{2,tx} + (u_1 \eta_2 + u_1 + u_{1,xx})_{xx} + u_{1,yy} = 0. \end{aligned} \tag{9}$$

Since (u_1, η_2) satisfies (4b), it follows that (9) reduces to

$$\begin{aligned} \eta_{tx} + (u\eta + u + u_{xx})_{xx} + u_{yy} = \partial_x \left\{ \frac{4w_x^2}{w^3} (w_t + \mu w_{xx} + u_1 w_x) \right. \\ - \frac{2w_{xx}}{w^2} (w_t + \mu w_{xx} + u_1 w_x) - \frac{4w_x}{w^2} \partial_x (w_t + \mu w_{xx} + u_1 w_x) \\ - \frac{2\mu}{w^2} [(\eta_2 + 1 - \mu u_{1,x}) w_x^2 + w_y^2] + \frac{2}{w} \partial_x^2 (w_t + \mu w_{xx} + u_1 w_x) \\ \left. + \frac{2\mu}{w} [(\eta_2 + 1) w_{xx} + \eta_{2,x} w_x + w_{yy} - \mu u_{1,xx} w_x] \right\} = 0. \end{aligned} \quad (10)$$

From (8) we get

$$w_t + \mu w_{xx} + u_1 w_x = wq(t, y) \quad (11)$$

where $q(t, y)$ is an arbitrary smooth function of t and y . The substitution of (11) into (10) leads to

$$-w^2 h(t, y) + w[(\eta_2 + 1)w_{xx} + \eta_{2,x}w_x + w_{yy} - \mu u_{1,xx}w_x] - [(\eta_2 + 1 - \mu u_{1,x})w_x^2 + w_y^2] = 0 \quad (12)$$

where $h(t, y)$ is also an arbitrary smooth function of t and y .

Therefore from the above calculation we have the conclusion:

Proposition. Expression (6) is an auto-Bäcklund transformation of (2+1)-dimensional Eckhaus-type extension of the dispersive long wave equation (4), where (u_1, η_2) is a solution of (4), and $w(x, y, t)$, $u_1(x, y, t)$, $\eta_2(x, y, t)$ satisfy (11) (or (8)) and (12) (or (10)).

Remark 1. Let $q = h = 0$ in (11) and (12). Then we get the overdetermined set of linear partial differential equations from (11) and (12)

$$\begin{cases} w_t + \mu w_{xx} + u_1 w_x = 0 \\ (\eta_2 + 1)w_{xx} + w_{yy} + (\eta_{2,x} - \mu u_{1,xx})w_x = 0 \\ w_y \mp w_x \sqrt{\mu u_{1,x} - \eta_2 - 1} = 0 \quad \mu u_{1,x} - \eta_2 - 1 > 0 \end{cases} \quad (13)$$

where (u_1, η_2) is some known solution of (4).

Remark 2. If we take the special solution of (4) as $(u_1 = 0, \eta_2 = c)$ and $q = h = 0$, then the auto-Bäcklund transformation (6) with (11) and (12) reduces to

$$u(x, y, t) = 2\mu \partial_x \log[w(x, y, t)] \quad \eta(x, y, t) = 2\partial_x^2 \log[w(x, y, t)] + c \quad (14)$$

where $w(x, y, t)$ satisfies the overdetermined set of linear partial differential equations

$$w_t + \mu w_{xx} = 0 \quad (15a)$$

$$(c + 1)w_{xx} + w_{yy} = 0 \quad (15b)$$

$$w_y \pm w_x \sqrt{-(c + 1)} = 0 \quad c + 1 < 0. \quad (15c)$$

Remark 3. (i) Equation (15a) is the well-known linear heat equation. (ii) Equation (15b) is the second-order linear wave equation. (iii) Equation (15c) is a first-order linear wave equation.

3. Explicit and exact solutions of the system (4)

In the following we would like to extract some physical exact solutions of (4) by studying the above-mentioned auto-Bäcklund transformation.

Type 1. When we take the initial solution of (4) as $[u_1 = c_1y + c_2, \eta_2 = \eta_2(t, y)]$, since the function w of every term in (11) or (12) has the same order, we assume that $w(x, y, t)$ is of the form

$$w(x, y, t) = P(t, y) + \exp[\Theta(t, y)x + \Psi(t, y)] \tag{16}$$

where $P(t, y) \neq 0, \Theta(t, y) \neq 0, \Psi(t, y)$ are functions to be determined.

With the aid of Maple, substituting (16) into (11) and (12) and equating the coefficients of these $x^i \exp[j(\Theta x + \Psi)]$ leads to the following set of partial differential equations:

$$\begin{cases} \Theta_t = \Theta_y = 0 \\ \Psi_t + \mu\Theta^2 + u_1\Theta = q \\ P_t = Pq \\ \Psi_{yy} = h \\ P(\eta_2 + 1)\Theta^2 + P\Psi_y^2 - 2P_y\Psi_y + P_{yy} = hP \\ -P_y^2 + PP_{yy} = hP^2. \end{cases} \tag{17}$$

Therefore we have exact solutions of (4) as:

Family 1. When $P(t, y) > 0$, we have

$$u(x, y, t) = \mu\Theta \tanh\left(\frac{1}{2}[\Theta x + \Psi(t, y) - \log P(t, y)]\right) + u_1 + \mu\Theta \tag{18}$$

$$\eta(x, y, t) = \frac{1}{2}\Theta^2 \operatorname{sech}^2\left(\frac{1}{2}[\Theta x + \Psi(t, y) - \log P(t, y)]\right) + \eta_2(t, y) \tag{19}$$

where $\Theta \neq 0$ is a constant, and $\Psi(t, y), P(t, y)$ satisfy (17).

Family 2. When $P(t, y) < 0$, we have

$$u(x, y, t) = \mu\Theta \coth\left(\frac{1}{2}[\Theta x + \Psi(t, y) - \log |P(t, y)|]\right) + u_1 + \mu\Theta \tag{20}$$

$$\eta(x, y, t) = \frac{1}{2}\Theta^2 \operatorname{cosech}^2\left(\frac{1}{2}[\Theta x + \Psi(t, y) - \log |P(t, y)|]\right) + \eta_2(t, y) \tag{21}$$

where $\Theta \neq 0$ is a constant, and $\Psi(t, y), P(t, y)$ satisfy (17).

In the following we consider only the case $P(t, y) > 0$. For example, from (17) we have

$$\begin{aligned} P(t, y) = p = \text{const.} \quad q(t, y) = h(t, y) = 0 \quad \Theta(t, y) = \theta = \text{const.} \\ \Psi(t, y) = \theta(-c_1t + c_3)y - (c_2\theta + \mu\theta^2)t \quad \eta_2(t, y) = -\frac{1}{p}(-c_1t + c_3)^2 - 1 \end{aligned} \tag{22}$$

where c_1, c_2, c_3 are all arbitrary constants.

Therefore when $p > 0$, we have the exact solution of (4)

$$u = \mu\theta \tanh\frac{1}{2}[\theta x + \theta(-c_1t + c_3)y - (c_2\theta + \mu\theta^2)t - \log p] + c_1y + c_2 + \mu\theta \tag{23a}$$

$$\eta = \frac{1}{2}\theta^2 \operatorname{sech}^2\frac{1}{2}[\theta x + \theta(-c_1t + c_3)y - (c_2\theta + \mu\theta^2)t - \log p] - \frac{1}{p}(-c_1t + c_3)^2 - 1. \tag{23b}$$

It follows that when $c_1 \neq 0$, the solution (23) is a nontravelling wave solution of (4), while when $c_1 = 0$, we have the travelling wave solution of (4)

$$u = \mu\theta \tanh \frac{1}{2}[\theta x + \theta c_3 y - (c_2\theta + \mu\theta^2)t - \log p] + c_2 + \mu\theta \tag{24a}$$

$$\eta = \frac{1}{2}\theta^2 \operatorname{sech}^2 \frac{1}{2}[\theta x + \theta c_3 y - (c_2\theta + \mu\theta^2)t - \log p] - \frac{c_3^2}{p} - 1. \tag{24b}$$

Remark 4. Let $z = \theta x + \theta c_3 y - (c_2\theta + \mu\theta^2)t - \log p$. Then when $z \rightarrow \pm\infty$, we have that $u \rightarrow (\pm 1 + 1)\mu\theta + c_2 = \text{const}$ and $\eta \rightarrow -\frac{c_3^2}{p} - 1 = \text{const}$.

Type 2. When $(u_1 = \text{const}, \eta_2 = \text{const} < -1)$, which is a trivial solution of (4), from (13) we can determine

$$w(x, y, t) = a_0 + \sum_{j=1}^N a_j \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j] \tag{25}$$

where $\eta_2 + 1 < 0$, e_j, θ_j and a_j are constants and $a_0 a_j \theta_j \neq 0, \theta_i \neq \theta_j (i \neq j)$.

Therefore we have exact solutions of (4)

$$u = \frac{2\mu \sum_{j=1}^N a_j \theta_j \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j]}{a_0 + \sum_{j=1}^N a_j \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j]} + u_1 \tag{26a}$$

$$\eta = \frac{2 \sum_{j=1}^N a_j \theta_j^2 \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j]}{a_0 + \sum_{j=1}^N a_j \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j]} - \frac{(\sum_{j=1}^N a_j \theta_j \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j])^2}{(a_0 + \sum_{j=1}^N a_j \exp [\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j])^2} + \eta_2. \tag{26b}$$

In particular, when $N = 1$, from (26a, b) we get the travelling wave solution of (4) which is equivalent to (24a, b). When $N = 2$, we derive the exact solution of (4)

$$u = \frac{2\mu(a_1\theta_1 e^{\xi_1} + a_2\theta_2 e^{\xi_2})}{a_0 + a_1 e^{\xi_1} + a_2 e^{\xi_2}} + u_1 \tag{27a}$$

$$\eta = \frac{2[a_0 a_1 \theta_1^2 e^{\xi_1} + a_0 a_2 \theta_2^2 e^{\xi_2} + a_1 a_2 (\theta_1 - \theta_2)^2 e^{\xi_1 + \xi_2}]}{[a_0 + a_1 e^{\xi_1} + a_2 e^{\xi_2}]^2} + \eta_2 \tag{27b}$$

where $\xi_j = \theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1\theta_j + \mu\theta_j^2)t + e_j (j = 1, 2), a_0 a_1 a_2 \theta_1 \theta_2 \neq 0, \theta_1 \neq \theta_2$.

Type 3. When $u_1 = 0, \eta_2 = \text{const} (\eta_2 + 1 < 0)$, following our idea [19], we know that $w(x, y, t)$ in (15) has the polynomial solution in the form

$$w = \sum_{i=0}^n \left[(-\mu)^{(n-i)} (n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n+1-i)} \frac{A_i (x \pm \sqrt{-(\eta_2 + 1)}y)^{2(n+1-i)-j}}{(2(n+1-i)-j)!} \right] (t - t_0)^i \tag{28}$$

where A'_i, t_0 are constants.

Therefore we have infinitely many rational solutions of (4)

$$u = \frac{2\mu \sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n-i)+1} \frac{A_i(x \pm \sqrt{-(\eta_2+1)y})^{2(n-i)+1-j}}{(2(n-i)+1-j)!} \right] (t-t_0)^i}{\sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n+1-i)} \frac{A_i(x \pm \sqrt{-(\eta_2+1)y})^{2(n+1-i)-j}}{(2(n+1-i)-j)!} \right] (t-t_0)^i} \quad (29a)$$

$$\eta = \frac{2 \sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n-i)} \frac{A_i(x \pm \sqrt{-(\eta_2+1)y})^{2(n-i)-j}}{(2(n-i)-j)!} \right] (t-t_0)^i}{\sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n+1-i)} \frac{A_i(x \pm \sqrt{-(\eta_2+1)y})^{2(n+1-i)-j}}{(2(n+1-i)-j)!} \right] (t-t_0)^i} - 2 \left(\frac{\sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n-i)+1} \frac{A_i(x \pm \sqrt{-(\eta_2+1)y})^{2(n-i)+1-j}}{(2(n-i)+1-j)!} \right] (t-t_0)^i}{\sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \sum_{j=1}^{2(n+1-i)} \frac{A_i(x \pm \sqrt{-(\eta_2+1)y})^{2(n+1-i)-j}}{(2(n+1-i)-j)!} \right] (t-t_0)^i} \right)^2 + \eta_2. \quad (29b)$$

In particular, when $n = 1$, we have from (34) the rational solution of (4) as

$$u = \frac{2\mu[A_1(-\mu z + t - t_0) + A_2]}{(A_1 z + A_2)(t - t_0) - \mu(\frac{1}{2}A_1 z^2 + A_2 z + A_3)} \quad (30a)$$

$$\eta = \frac{2[A_1^2(t - t_0)^2 + (\mu A_1^2 z - A_1 A_2(2 + \mu))(t - t_0) + R]}{[(A_1 z + A_2)(t - t_0) - \mu(\frac{1}{2}A_1 z^2 + A_2 z + A_3)]^2} + \eta_2 \quad (30b)$$

where A_1, A_2, A_3, t_0 are constants, $z = x \pm y\sqrt{-(\eta_2 + 1)}$, $R = -\frac{1}{2}A_1^2 z^2 + A_1 A_2(2\mu + 1)z + A_1 A_3 - A_2^2$. The physical meaning of these rational solutions needs to be explored further.

Type 4. When $u_1 = \text{const}$, $\eta_2 = \text{const}$ ($\eta_2 + 1 < 0$), we know that $w(x, y, t)$ in (15) has the solution in the form

$$w = \sum_{j=1}^N b_j \exp[\theta_j x \pm y\theta_j \sqrt{-(\eta_2 + 1)} - (u_1 \theta_j + \mu \theta_j^2)t] + \sum_{i=0}^n \left[(-\mu)^{(n-i)}(n-i)! \binom{n}{n-i} \times \sum_{j=1}^{2(n+1-i)} \frac{B_i(x \pm \sqrt{-(\eta_2 + 1)y - u_1 t})^{2(n+1-i)-j}}{(2(n+1-i)-j)!} \right] t^i \quad (31)$$

where b'_i, B'_i are all constant.

Therefore we have infinitely many exact solutions of (4)

$$u = 2\mu w_x/w + u_1 \quad \eta = 2w_{xx}/w - 2w_x^2/w^2 + \eta_2 \quad (32)$$

where

$$w_x = \sum_{j=1}^N b_j \theta_j \exp [\theta_j x \pm y \theta_j \sqrt{-(\eta_2 + 1)} - (u_1 \theta_j + \mu \theta_j^2) t] \\ + \sum_{i=0}^n \left[(-\mu)^{(n-i)} (n-i)! \binom{n}{n-i} \right. \\ \left. \times \sum_{j=1}^{2(n-i)+1} \frac{B_i(x \pm \sqrt{-(\eta_2 + 1)} y - u_1 t)^{2(n-i)+1-j}}{(2(n-i) + 1 - j)!} \right] t^i \quad (33)$$

$$w_{xx} = \sum_{j=1}^N b_j \theta_j^2 \exp [\theta_j x \pm y \theta_j \sqrt{-(\eta_2 + 1)} - (u_1 \theta_j + \mu \theta_j^2) t] \\ + \sum_{i=0}^n \left[(-\mu)^{(n-i)} (n-i)! \binom{n}{n-i} \right. \\ \left. \times \sum_{j=1}^{2(n-i)} \frac{B_i(x \pm \sqrt{-(\eta_2 + 1)} y - u_1 t)^{2(n-i)-j}}{(2(n-i) - j)!} \right] t^i \quad (34)$$

Particularly, when $N = n = 1$, we have the nontravelling wave solution of (4)

$$u = \mu \left[\theta_1 - \frac{g(x, y, t)}{f(x, y, t)} \right] \tanh \frac{1}{2} [\xi_1 - \log f(x, y, t)] + u_1 \quad (35a)$$

$$\eta = \frac{1}{2} \left[\theta_1 - \frac{g(x, y, t)}{f(x, y, t)} \right] \left[\theta_1 - \frac{f_x(x, y, t)}{f(x, y, t)} \right] \operatorname{sech}^2 \frac{1}{2} [\xi_1 - \log f(x, y, t)] \\ + \frac{f_x(x, y, t)g(x, y, t) - f(x, y, t)g_x(x, y, t)}{f^2(x, y, t)} \tanh \frac{1}{2} [\xi_1 - \log f(x, y, t)] + \eta_2 \quad (35b)$$

where

$$g(x, y, t) = B_1[-\mu(x \pm y\sqrt{-(\eta_2 + 1)} - u_1 t) + t] + B_2 \\ f(x, y, t) = [B_1(x \pm y\sqrt{-(\eta_2 + 1)} - u_1 t) + B_2]t - \mu \left[\frac{1}{2} B_1(x \pm y\sqrt{-(\eta_2 + 1)} - u_1 t)^2 \right. \\ \left. + B_2(x \pm y\sqrt{-(\eta_2 + 1)} - u_1 t) + B_3 \right] > 0 \\ \xi_1 = \theta_1 x \pm y \theta_1 \sqrt{-(\eta_2 + 1)} - (u_1 \theta_1 + \mu \theta_1^2) t.$$

When $f(x, y, t) < 0$, we have another solution of (4)

$$u = \mu \left[\theta_1 - \frac{g(x, y, t)}{f(x, y, t)} \right] \coth \frac{1}{2} [\xi_1 - \log |f(x, y, t)|] + u_1 \quad (36a)$$

$$\eta = \frac{1}{2} \left[\theta_1 - \frac{g(x, y, t)}{f(x, y, t)} \right] \left[\theta_1 - \frac{f_x(x, y, t)}{f(x, y, t)} \right] \operatorname{cosech}^2 \frac{1}{2} [\xi_1 - \log |f(x, y, t)|] \\ + \frac{f_x(x, y, t)g(x, y, t) - f(x, y, t)g_x(x, y, t)}{f^2(x, y, t)} \\ \times \coth \frac{1}{2} [\xi_1 - \log |f(x, y, t)|] + \eta_2. \quad (36b)$$

Remark 5. The above-mentioned solutions have been tested for correctness via Maple software.

4. Conclusions and discussions

In summary, we have obtained an auto-Bäcklund transformation by truncating the Laurent series expansion at $O(w^0)$, though the (2+1)-dimensional Eckhaus-type extension of the dispersive long wave equation (4) has no Painlevé property. Moreover many types of exact solutions have been found through further considering the above-mentioned Bäcklund transformation. These obtained families of solutions may be of important significance in explaining some physical phenomena related to system (4). It should be pointed out that other types of exact solutions of (4) may also be obtained using the obtained auto-Bäcklund transformation (6) and other ansätze. Moreover it is also important to seek other useful transformations to extract more types of solutions of (4). For example, if we introduce the Hirota bilinear operators D_t, D_x, D_y as follows [20]

$$D_t^m D_x^n D_y^k F \cdot G = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^k \\ \times F(x, y, t) G(x', y', t')|_{t=t', x=x', y=y'} \quad (37)$$

then under the transformation (6), from (8) and (10) we can change system (4) into the following bilinear form

$$(D_t + u_1 D_x + u_{1,x} + D_x^2) w_x \cdot w = 0 \quad (38a)$$

$$[(\eta_2 + 1) D_x + \eta_{2,x} - \mu u_{1,x} + \mu u_{1,x} D_x] w_x \cdot w + D_y w_y \cdot w = 0 \quad (38b)$$

where (u_1, η_2) is a solution of (4), which may be used to seek multi-soliton solutions of (4). These will be further considered.

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